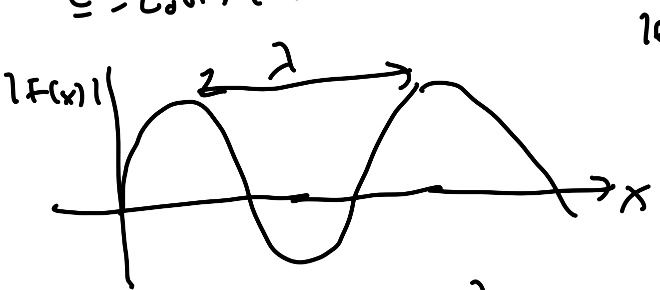
Energy and momentum of EM waves

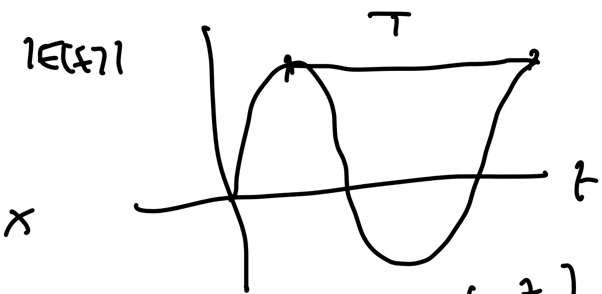
Last time:

* EM waves → transverse → $\underline{E} \perp \underline{B} \perp \hat{k}$
  → propagate at c 3×10^8 m/s
  → Magnitudes of $\underline{E}$ & $\underline{B}$;
  $E_0 = \pm cB_0$
  → monochromatic

* wave:
  $\underline{E} = E_0 \sin(kx - \omega t)$



$\underline{E} = E_0 \sin(k_x x)$



$\underline{E} = E_0 \sin(\omega t)$

$\omega = \frac{2\pi}{T} = 2\pi\nu$; $\omega = ck$; $\lambda\nu = c$; $\nu \equiv$ frequency

* Polarization → linearly polarized → $\underline{E}_0$ & $\underline{B}_0$ directions are constant in time.
  → circularly polarized → $\underline{E}_0$ & $\underline{B}_0$ directions rotate with a freq ω.

Electromagnetic energy

Suppose we have charge & current configuration
$\downarrow$ $\underline{E}$ $\downarrow$ $\underline{B}$

Now suppose the charge moves around in time.

How much work dW is done by EM forces acting on these charges at the time interval dt?

$$dW = \underline{F}\cdot d\underline{l} = q(\underline{E} + \underline{v}\times\underline{B})\cdot\underline{v}\,dt = q\underline{E}\cdot\underline{v}\,dt + \cancel{\underline{v}\times\underline{B}\cdot\underline{v}}^{\,0}\,dt$$
$$= q\underline{E}\cdot\underline{v}\,dt \quad (*)$$

we'll rewrite this in terms of q and $\rho\underline{v}$

$q \to \int \rho\, d\tau$

$\rho\underline{v} \to \underline{J}$ with this eq (*) becomes:

$$dW = \rho\, d\tau\, \underline{E} \cdot \frac{\underline{J}}{\rho} dt = \underline{E} \cdot \underline{J}\, d\tau\, dt \Rightarrow \frac{dW}{dt} = \int_V \underbrace{(\underline{E} \cdot \underline{J})}_{\text{work per unit time per unit volume.}} d\tau$$

To rewrite this use Maxwell's eq:

$$\nabla \times \underline{B} = \mu_0 \underline{J} + \mu_0 \varepsilon_0 \frac{\partial \underline{E}}{\partial t}$$

$$\Rightarrow \underline{J} = \frac{\nabla \times \underline{B}}{\mu_0} - \frac{\mu_0 \varepsilon_0}{\mu_0} \frac{\partial \underline{E}}{\partial t}$$

We can write $\underline{E} \cdot \underline{J}$ as:

$$\underline{E} \cdot \underline{J} = \frac{1}{\mu_0} \underline{E} \cdot (\nabla \times \underline{B}) - \varepsilon_0 \underline{E} \cdot \frac{\partial \underline{E}}{\partial t} \;**$$

we can rewrite this using product rules:

$$\nabla \cdot (\underline{E} \times \underline{B}) = \underline{B} \cdot (\nabla \times \underline{E}) - \underline{E} \cdot (\nabla \times \underline{B}) = -\underline{B} \cdot \frac{\partial \underline{B}}{\partial t} - \underline{E} \cdot (\nabla \times \underline{B})$$

....

$$\underline{E} \cdot \underline{J} = -\frac{1}{\mu_0} \nabla \cdot (\underline{E} \times \underline{B}) - \frac{1}{\mu_0} \underline{B} \cdot \frac{\partial \underline{B}}{\partial t} - \varepsilon_0 \underline{E} \cdot \frac{\partial \underline{E}}{\partial t}$$

$$\Rightarrow \underline{E} \cdot \underline{J} = -\frac{1}{2} \frac{\partial}{\partial t} \underbrace{\left(\varepsilon_0 E^2 + \frac{1}{\mu_0} B^2 \right)} - \frac{1}{\mu_0} \nabla \cdot (\underline{E} \times \underline{B})$$

Now subs this into the equation for $\frac{dW}{dt}$:

$$\frac{dW}{dt} = \int_V (\underline{E} \cdot \underline{J})\, d\tau = -\frac{d}{dt} \int_V \frac{1}{2} \left(\varepsilon_0 E^2 + \frac{1}{\mu_0} B^2 \right) d\tau$$

$$- \boxed{\frac{1}{\mu_0} \int_V \nabla \cdot (\underline{E} \times \underline{B})\, d\tau} = \oint_S (\underline{E} \times \underline{B}) \cdot d\underline{a}$$

$$\boxed{\underbrace{\frac{dW}{dt}}_{\text{work/unit time}} = -\frac{d}{dt} \int_V \frac{1}{2} \underbrace{\left(\varepsilon_0 E^2 + \frac{1}{\mu_0} B^2 \right) d\tau}_{\text{energy density stored in the fields}} - \frac{1}{\mu_0} \underbrace{\oint_S (\underline{E} \times \underline{B}) \cdot d\underline{a}}_{\text{Flux of energy.}}}$$

Poynting's theorem

Work-energy theorem for electrodynamics

$$W_e = \frac{\varepsilon_0}{2} \int E^2 d\tau$$

$$W_m = \frac{1}{2\mu_0} \int B^2 d\tau$$

$$\boxed{\text{work done on the charges by the EM force} = \text{decrease in energy stored in the fields} - \text{energy that flows out of the surface}}$$

The energy per unit time, per unit area, transported by the fields is

$$\boxed{\underline{S} = \frac{1}{\mu_0}(\underline{E} \times \underline{B})}$$ Poynting vector

Specifically $\underline{S} \cdot d\underline{a}$ is the energy per unit time crossing the infinitesimal surface $d\underline{a}$ → we can interpret $\underline{S}$ as the energy flux density.

We can rewrite the eq with these definitions

$$\boxed{\frac{dW}{dt} = -\frac{d}{dt}\int_V u \, d\tau - \oint_S \underline{S} \cdot d\underline{a}}$$

In an empty region of space (no charge $\frac{dW}{dt} = 0$)

$$\frac{d}{dt}\int_V u \, d\tau = \oint_S \underline{S} \cdot d\underline{a}$$

$$\Rightarrow \int_V \frac{du}{dt} d\tau \underset{\text{div theo}}{=} \int_V (\nabla \cdot \underline{S}) \, d\tau$$

$$\Rightarrow \frac{\partial u}{\partial t} = -\nabla \cdot \underline{S}$$ ← looks like the continuity equation. statement of local conservation of energy.

Thoughts on the Poynting vector

$$\frac{\partial u}{\partial t} = -\nabla \cdot \underline{S} \Rightarrow \frac{\partial V}{\partial t} = -\int_V \nabla \cdot \underline{S} \, d\tau = -\int_{\partial A} \underline{S} \cdot d\underline{a} = -\Phi_S(A)$$

- rate of change of EM energy in volume V is given by the flux of the Poynting vector $\underline{S}$ through a surface A.

If dA points outward ⇒ V increases when $\underline{S}$ is opposite to dA

* $\underline{S} = \frac{1}{\mu_0} \underline{E} \times \underline{B}$ points in the direction of flow of EM energy.

* $|\underline{S}| = \frac{1}{\mu_0} |\underline{E} \times \underline{B}| = \frac{EB}{\mu_0}$ (since $E \perp B$ so $\sin\theta = 1$) magnitude of S is known as the intensity.

* Units of S

$$[S] = \frac{[E][B]}{[\mu_0]} = \frac{\left(\frac{N}{C}\right)(T)}{\frac{N}{A^2}} = \frac{kg \, m^2 s^{-3}}{m^2} = \frac{\text{Watts}}{m^2}$$ ← power ← area

Example: suppose the electric and magnetic components of a plane EM wave propagating in the +x direction are:

$$\underline{E} = E_0 \cos(kx - \omega t)\hat{j}$$

$$\underline{B} = B_0 \cos(kx - \omega t)\hat{k}$$

What is the Poynting vector?

$\underline{S} = \frac{1}{\mu_0}(\underline{E} \times \underline{B})$ in this case

$$\underline{S} = \frac{1}{\mu_0}\left[E_0 \cos(kx-\omega t)\hat{j}\right) \times \left[B_0 \cos(kx - \omega t)\hat{k}\right]$$

$$= \frac{E_0 B_0}{\mu_0} \cos^2(kx - \omega t)\hat{i}$$

← as expected $\underline{S}$ points in the direction of wave propagation.

We can also calculate the average intensity

$$I = \langle |S| \rangle = \frac{E_0 B_0}{\mu_0} \langle \underbrace{\cos^2(kx-\omega t)}_{1/2} \rangle = \frac{E_0 B_0}{2\mu_0} = \frac{E_0^2}{2c\mu_0} = \frac{cB_0^2}{2\mu_0}$$

we can relate to intensity & the energy density in the electric & magnetic field.

$$u_B = \frac{B^2}{2\mu_0} \underset{E = \frac{B}{c}}{=} \frac{(E/c)^2}{2\mu_0} = \frac{E^2}{2c^2\mu_0} = \frac{\varepsilon_0 E^2}{2} = u_E$$

The average ~~total~~ energy density:

$$\langle u \rangle = \langle u_E + u_B \rangle = \left\langle 2\left(\frac{\varepsilon_0 E^2}{2}\right)\right\rangle = \varepsilon_0 \langle E^2 \rangle = \frac{\varepsilon_0}{2} E_0^2 \quad (**)$$

we can write the intensity in ~~the~~ terms of this av energy density:

$$\boxed{I = \langle S \rangle c \langle u \rangle}$$

Example: The magnitude of the Poynting vector at the surface of the earth is referred to as the solar constant

$$\langle S \rangle = 1.35 \times 10^3 \; W/m^2$$

a) Assume that the sun's EM radiation is a plane sinusoidal wave. What are the magnitudes of E & B?

b) What is the total time averaged power radiated by the sun? We will consider the earth-sun distance $= R = 1.5 \times 10^{11}$ m

$$\langle S \rangle = c\langle u \rangle = c\left(\frac{\epsilon_0}{2}E_0^2\right) \Rightarrow E_0 = \left[\frac{2\langle S\rangle}{c\epsilon_0}\right]^{1/2}$$

After this we can calculate B using $B_0 = \frac{E_0}{c}$

Plugging in the numbers:

$$E_0 = \left(\frac{2\langle S\rangle}{c\epsilon_0}\right)^{1/2} = 1.01\times10^{3}\,V/m$$

$$B_0 = \frac{E_0}{c} = 3.4\times10^{-6}\,T$$

Momentum & radiation pressure.

EM waves carry energy they can exert radiation pressure.

If a plane EM wave is completely absorbed the momentum transferred is

$$\boxed{\Delta p = \frac{\Delta U}{c}}$$ complete absorption

$p \equiv$ momentum
$U \equiv$ energy
$c \equiv$ speed of light.

If an EM wave is completely reflected by a surface:

$$\boxed{\Delta p = 2\frac{\Delta U}{c}}$$ complete reflection.

For the complete absorption the average radiation pressure (Force/unit area) is given by:

$$P = \frac{\langle F\rangle}{A} = \frac{1}{A}\left\langle\frac{dp}{dt}\right\rangle = \frac{1}{Ac}\left\langle\frac{dU}{dt}\right\rangle$$

P = radiation pressure
F = force
A = area.

And the rate of energy delivered to the surface:

$$\left\langle\frac{dU}{dt}\right\rangle = \langle S\rangle A = IA$$

intensity.

We can write the radiation pressure as:

$$\boxed{P = \frac{I}{c}}$$ complete absorption

Complete reflection

$$\boxed{P = \frac{2I}{c}}$$

complete reflection.

How do we actually make an EM wave?

EM waves propagate at the speed of light in empty space.

How do we generate an EM wave?

We will discuss a simple planar geometry.

For a wave on a string → "shake" the string to make a wave
- → we have to do work against the tension in the string
- → The work we put into the system is carried off as an energy flux.

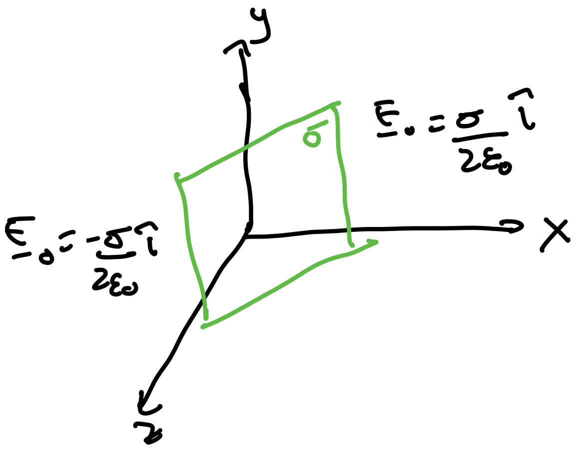
What is the analogy for EM waves?

→ We can think of $\underline{E}$ as the "string".

→ There is a "tension" associated with electric field lines.
- → there is a force that resists the "shaking"

→ Then a wave will propagate along a field line as a result of the "shaking".

Suppose we have a sheet of charge located in the yz plane.
The sheet has a charge density σ:

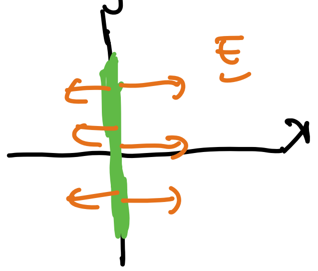


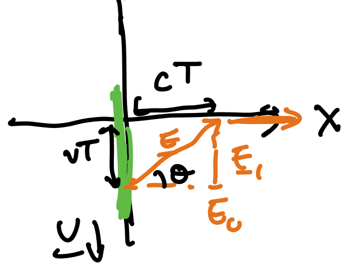
If the sheet is at rest, the surface will give rise to an $\underline{E}$ field

$$\underline{E}_0 = \begin{cases} \sigma/2\varepsilon_0\,\hat{i}\,; & x>0 \\ -\sigma/2\varepsilon_0\,\hat{i}\,; & x<0 \end{cases}$$

Now at $t=0$ we start to pull on the sheet with a constant velocity $\underline{v} = -v\hat{j}$

What happens with the field at time $t=T$?





at $t < 0$
$\underline{v} = 0$
$$\underline{E}_0 = \frac{\pm\sigma}{2\varepsilon_0}\hat{i}$$

at $t = T$
$\underline{v} = -v\hat{j}$

What happens to $\underline{E}$ generated by the charge on a fixed point in the sheet, as we pull the sheet downwards at $t=0$

The point at $y=0$ will now be at $y=-vT$ at $t=T$

If information propagates at the speed of light c, the portion of the field i.e. located at $x > cT$ "doesn't know" charges are moving.

The new field will be
$$\underline{E} = \underline{E}_0 + \underline{E}_1$$
And from the schematic:
$$\tan\theta = \frac{E_1}{E_0} = \frac{vT}{cT} = \frac{v}{c}$$
where $E_1 = |\underline{E}_1|$; $E_0 = |\underline{E}_0|$
θ is the ∡ with the x axis

Remember $E_0 = \sigma/2\varepsilon_0$ we can write E_1 in terms of E_0:
$$E_1 = \left(\frac{v}{c}E_0\right)\hat{j} = \left(\frac{v\sigma}{2\varepsilon_0 c}\right)\hat{j}$$
, $\underline{E}_1$ is the perturbation to the static field due to the movement of the sheet

Direction of E_1 is such that the forces it exerts on the charges on the sheet resist the motion of the sheet.

For an infinitesimal area dA of the sheet with charge $dq = \sigma dA$, the "upward tension" associated with the electric field is:
$$d\underline{F}_e = dq\,\underline{E}_1 = (\sigma dA)\left(\frac{v\sigma}{2\varepsilon_0 c}\right)\hat{j} = \left(\frac{v\sigma^2 dA}{2\varepsilon_0 c}\right)\hat{j}$$
The external force $\underline{F}_{ext}$ that needs to be applied to overcome this tension is
$$d\underline{F}_{ext} = -d\underline{F}_e = -\left(\frac{v\sigma^2 dA}{2\varepsilon_0 c}\right)\hat{j}$$

and $\quad dW_{ext} = \underline{F}_{ext}\cdot d\underline{s}$

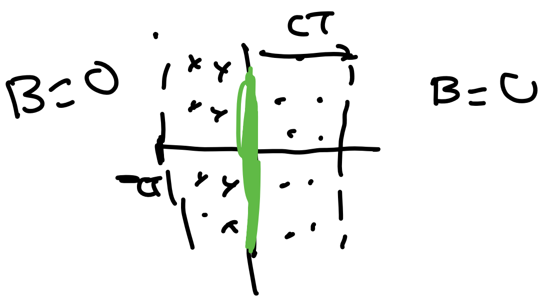
The work per unit time per unit area exerted externally is:

$$\frac{d^2 W_{ext}}{dA\,dt} = \frac{d\underline{F}_{ext} \cdot d\underline{s}}{dA\,dt} = \frac{d\underline{F}_{ext}}{dA} \cdot \frac{d\underline{s}}{dt}$$

$$= \left(-\frac{V\sigma^2}{2\varepsilon_0 c}\hat{j}\right) \cdot \left(-V\hat{j}\right) = \frac{V^2\sigma^2}{2\varepsilon_0 c}$$

In the process of moving the sheet is created a current $\underline{K} = -\sigma V\hat{j}$. From Ampère's law we know that this will create a magnetic field:

$$\underline{B}_1 = \begin{cases} (\mu_0 \sigma V/2)\hat{k} & ; x > 0 \\ -(\mu_0 \sigma V/2)\hat{k} & ; x < 0 \end{cases}$$



The magnetic field $\underline{B}_1$ generated by the current is $\perp \underline{E}_1$ & has a magnitude $B_1 = \frac{E_1}{c}$ as expected.

What is the energy carried away by B_1 & E_1?

The energy flux is given by the Poynting vector:

$$\underline{S} = \frac{1}{\mu_0}\underline{E}_1 \times \underline{B}_1 = \frac{1}{\mu_0}\left(\frac{V\sigma}{2\varepsilon_0 c}\hat{j}\right) \times \left(\frac{\mu_0 \sigma V}{2}\hat{k}\right)$$

$$= \left(\frac{V^2\sigma^2}{4\varepsilon_0 c}\right)\hat{i}$$

$\therefore$ the total energy flux carried off by the perturbation $\underline{E}_1$ & $\underline{B}_1$ is equal to the rate of work/unit area to pull the charged sheet down against the tension in $\underline{E}_1$

To generate a sinusoidal wave with angular frequency ω we can "pull" the sheet of charge up & down at vel $\underline{v}(t) = -v_0 \cos\omega t \, \hat{j}$
this oscillation will give the following fields:

$$\left.\begin{aligned} \underline{E}_1 &= \frac{c\mu_0 \sigma v_0}{2} \cos\omega\left(t - \frac{x}{c}\right)\hat{j} \\ \underline{B}_1 &= \frac{\mu_0 \sigma v_0}{2} \cos\omega\left(t - \frac{x}{c}\right)\hat{k} \end{aligned}\right\} \text{ for } x < 0$$

&

$$\left.\begin{aligned} \underline{E}_1 &= c\mu_0 \sigma v_0 \cos\omega\left(t + \frac{x}{c}\right)\hat{j} \\ \underline{B} &= \frac{-\mu_0 \sigma v_0}{2} \cos\omega\left(t + \frac{x}{c}\right)\hat{k} \end{aligned}\right\} \text{ for } x < 0$$